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Exam I, MTH 512, Spring 2015

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QUESTION 1. (i) Let $T : V \longrightarrow V$ be a linear transformation such that V is finite dimensional. Let $b \in R$. Prove that there is a number $a \in R$ such that $|a - b| < 0.001$ and $T - aI$ is one-to-one (injective).

sketch proof: Choose $d \in R$ such that $|d - b| < 0.001$. Since V is finite dimensional, T has finitely many eigenvalues. Since there are infinitely many numbers between d and b, there is a number, say a, such that a is not an eigenvalue of T. It is clear that $|a - b| < 0.001$. Since a is not an eigenvalue of T, we know that T is one-to-one.

(ii) Let $T : V \longrightarrow V$ be a linear transformation. Assume v, w are nonzero vectors in V such that $T(v) = 3w$ and $T(w) = 3v$. Prove that 3 or -3 is an eigenvalue of T.

sketch proof: It is clear that $T(v + w) = 3(v + w)$. If $v + w$ is not O_V, then 3 is an eigenvalue of T. Hence assume that $v + w = O_V$. Then $v = -w$. Thus $T(v) = T(-w) = -3v$. Thus -3 is an eigenvalue of T.

(iii) Let $T : V \longrightarrow V$ be a linear transformation. Assume that $u, v, u + v$ are eigenvectors of T. Prove that u and v are eigenvectors of T corresponding to the same eigenvalue of T.

sketch proof: Given $T(u + v) = a(u + v) = au + av, T(u) = a_1u$, and $T(v) = a_2v$ for some real numbers a, a_1, a_2. We show $a_1 = a_2$. Thus $T(u+v) = au+av = T(u)+T(v) = a_1u+a_2v$. Thus $(a-a_1)v+(a-a_2)u = 0_V$. We consider two cases. Case I. Suppose that u, v independent. Then $a - a_1 = a - a_2 = 0$. Thus $a_1 = a = a_2$ and we are done. Assume that u, v are dependent. Then $u = dv$ for some nonzero $d \in R$. Thus $a_1u = T(u) = T(dv) = da_2v = a_2dv = a_2u$. Hence $(a_1 - a_2)u = O_V$. Since u is a nonzero vector, $a_1 - a_2 = 0$. Thus $a_1 = a_2$.

(iv) Let A be an $n \times n$ matrix such that the sum of the entries in each row of A equals 4. Prove that 4 is an eigenvalue of A. Find an eigenvector of A that corresponds to the eigenvalue 4.

sketch proof: It is clear that $A \times \begin{bmatrix}1\\1\\.\\.\\.\\1\end{bmatrix} = \begin{bmatrix}4\\4\\.\\.\\.\\4\end{bmatrix} = 4\begin{bmatrix}1\\1\\.\\.\\.\\1\end{bmatrix}$**. Thus 4 is an eigenvalue of A.**

(v) Let A be a 4×4 matrix. Given $2, 3$ are eigenvalues of A such that $E_2 = \{(a, b, 0, 0)|a, b \in R\}$ and $E_3 = \{(0, 0, c, d)|c, d \in R\}$. Find $C_A(\alpha)$. Find the matrix A. If it is impossible to determine A, then explain.

comment: No ideas, just trivial and typical calculations... all of you got it right.

(vi) Let $T : P_3 \longrightarrow P_3$ be a linear transformation such that $T(x) = 4x$, $T(x^2) = 2x$, and $T(6) = 12$. Describe all elements in P_3 that have image $x + 1$ under T.

sketch proof: This particular question can be done by staring and simple observation. Observe $T(1) = 2$. Note that $Range(T) = span\{2, 2x, 4x\} = span\{1, 2x\}$. Thus $dim(Range(T)) = 2$. Since $dim(Ker(T)) + dim(Range(T)) = 3$, we conclude that $dim(Ker(T)) = 1$. Since $T(x) = T(2x^2)$, we have $T(x - 2x^2) = 0$. Thus $Ker(T) = span\{x-2x^2\}$.Now just find one element say v such that $T(v) = x+1$. By staring, $v = \frac{1}{2}x^2+\frac{1}{2}$ or you may select $v = \frac{1}{4}x + \frac{1}{2}$. Thus $\{d+v \mid d \in Ker(T)\}$ is the set of all elements in P_3 that have image $x + 1$ under T.

(vii) Let $T : P_3 \longrightarrow P_3$ such that $T(ax^2 + bx + c) = (a + 3b + c)x^2 + (2b + c)x + 3c$. Is A diagnolizable?. If yes, then find a diagonal matrix D and an invertible matrix W such that $W^{-1}AW = D$.

sketch proof: Typical question...all of you did the calculations

(viii) Let $T : P_3 \longrightarrow P_3$ be a linear transformation. Given $\sqrt{2}, \sqrt[3]{2}$, and $\sqrt[4]{2}$ are eigenvalues of T. Show that there must exist a polynomial $f(x) \in P_3$ such that $T(f(x)) - 7f(x) = \sqrt{2}x^2 + \sqrt[3]{2}x + \sqrt[4]{2}$

sketch proof: Since $dim(P_3) = 3$ and P_3 cannot have more than 3 eigenvalues, we conclude that 7 is not an eigenvalue of T. Hence we know that the linear transformation $K : P_3 \longrightarrow P_3$, where $K(L(x)) = T(L(x)) - 7I(L(x)) = T(L(x)) - 7L(x)$ for every $L(x) \in P_3$, is one-to-one, and thus is ONTO (surjective) (since P_3 is finite dimensional). Since K is onto and $w(x) = \sqrt{2}x^2 + \sqrt[3]{2}x + \sqrt[4]{2} \in Range(K) = P_3$, we conclude there is a polynomial $f(x) \in P_3$ such that $K(f(x)) = T(f(x)) - 7f(x) = w(x) = \sqrt{2}x^2 + \sqrt[3]{2}x + \sqrt[4]{2}$

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